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DIVISION BY A HOLOMORPHIC MATRIX IN STRICTLY PSEUDOCONVEX DOMAINS

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ABSTRACT

Let F and G be respectively a vector- and a matrix-function in a bounded strictly pseudoconvex domain D, with entries holomorphic in D and continuous in $\bar{D}$. We prove that if F can be divided locally by G with holomorphic factors in a neighborhood of a given point w in D, and the rank of G is maximal at all points of $\bar{D}\setminus\{w\}$, then the division of F by G holds globally, with some factors which are holomorphic in D and continuous in $\bar{D}$. This method applies also to other function algebras in pseudoconvex domains.

1. Introduction

In 1971 Øvrelid proved the following division theorem:

THEOREM ([19], Theorem 1). *Let D be a bounded strictly pseudoconvex domain in C^n with C^2 boundary. Fix $w \in D$. Suppose that the functions $g_1, \ldots, g_q \in A(D)$ (the space of functions holomorphic in D and continuous in $\bar{D}$) satisfy the following conditions*:

(i) *The only common zero of functions $g_1, \ldots, g_q$ in $\bar{D}$ is w.*

(ii) *The germs $(g_1)_w, \ldots, (g_q)_w$ of functions $g_1, \ldots, g_q$ at w generate the ideal of germs of holomorphic functions, vanishing at w.*

Then for every $f \in A(D)$ with $f(w) = 0$ there exist functions $h_1, \ldots, h_q \in A(D)$ such that $f = \sum_{i=1}^{q} g_i h_i$ in $\bar{D}$.

Many results of this kind, for various function spaces and in a different setting, were obtained by other authors—see e.g. [6], [15], [1], [14], [17], [18], [9], [11].

Actually the proof of Øvrelid's theorem gives the following more general "local-to-global" decomposition result:

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THEOREM. *Let D and w be as above. Suppose that the functions $g_1, \ldots, g_q$ and $f \in A(D)$ are such that there exists a neighborhood U of w in D and functionals $\tilde{h}_1, \ldots, \tilde{h}_q$ holomorphic in U with $f = \sum_{i=1}^{q} g_i \tilde{h}_i$ in U, and*

(1) *for every $z \in \bar{D} \setminus \{w\}$ there exists l such that $g_l(z) \neq 0$.*

Then $f = \sum_{i=1}^{q} g_i h_i$ in $\bar{D}$ with some functions $h_i \in A(D)$.

In this note we are interested in a matrix analogue of the above result. Suppose that the functions f_i and g_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, q$ are from the algebra $A(D)$. We wish to find conditions on f_i and g_{ij} which assure that there exist functions $h_1, \ldots, h_q$, also from the algebra $A(D)$, such that for every $i = 1, \ldots, p$,

(2) $$f_i = \sum_{j=1}^{q} g_{ij} h_j \qquad \text{in } \bar{D}.$$

Suppose that like in the previous theorem there exists a neighborhood U of a fixed point w in D such that (2) holds in U with some $\tilde{h}_1, \ldots, \tilde{h}_q$ holomorphic in U. On the other hand, the condition (1) for a point $z \in \bar{D} \setminus U$ assures the existence of the local decomposition $f = \sum_{i=1}^{q} g_i k_i$ for some functions $k_i \in A(D \cap V)$, where V is some neighborhood of z in $\bar{D}$. Therefore, looking for the sufficient conditions on the decomposition (2) in all of $\bar{D}$ it is natural to assume that for every z in $\bar{D} \setminus \{w\}$:

(3) The rank of the matrix $[g_{ij}(z)]$ is maximal.

We prove here the following "local-to-global" decomposition theorem with respect to the matrix:

THEOREM 1. *Suppose that D is a bounded strictly pseudoconvex domain in $\mathbf{C}^n$. Fix $w \in D$. Let the functions f_i, g_{ij} from the algebra $A(D)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, satisfy the following assumptions:*

There exists a neighborhood U of w in D and functions $\tilde{h}_1, \ldots, \tilde{h}_q$ holomorphic in U, such that the decomposition

(4) $$f_i = \sum_{j=1}^{q} g_{ij} \tilde{h}_j, \qquad i = 1, \ldots, p,$$

holds in U.

For every $z \in \bar{D} \setminus \{w\}$, the condition (3) *is satisfied.*

Then there exist functions $h_1, \ldots, h_q \in A(D)$ such that (2) *holds in the whole domain $\bar{D}$.*

The proof of Øvrelid's theorem is based on results on the solution of the $\bar{\partial}$-problem with bounds in strictly pseudoconvex domains and on some cohomological device, known as the Koszul complex. The application of this method to the proof of decomposition results (for $p = 1$) was introduced by Hörmander [7] for some rings of analytic functions satisfying growth conditions, and then involved by Kelleher and Taylor [13]. In this last paper the authors considered also the decomposition with respect to the matrix ($p > 1$) and obtained similar results as in the vector case (i.e. $p = 1$) by the use of the generalized Koszul complex, introduced by D. Buchsbaum in [4] (the details of the proofs were not published in [13]). The results on the "local-to-global" decomposition with respect to the matrix were obtained also by Siu [20] for functions with restricted growth and by Bierstone and Milman [2] for functions from the algebra $A^{\infty}(D)$ in weakly pseudoconvex smoothly bounded domains. The proofs of theorems from [20] and [2] are based also on the solution of the $\bar{\partial}$-problem with bounds and on the Koszul complex, but of different type than that used in [7], [13] or [19]. It seems that this different sort of Koszul complex is not directly applicable, e.g., for algebras $A(D)$. On the other hand, by the application of the generalized Koszul complex from [4] one can obtain the result from Theorem 1. In order to avoid algebraic difficulties, which arise by the use of the Buchsbaum complex, we propose here another method, whose cohomological argument is a generalization of that from [19]. It should be pointed out that this method is closely related to the Buchsbaum complex, e.g. the main formula in the proof of Lemma 1 is the first pull-back map in the Buchsbaum complex. However, by the use of our method we avoid constructing the other terms of the resolution of the matrix map G induced by the functions g_{ij}. It turns out that this method works also for some other function algebras, and in more general domains; we indicate this in the last section. We discuss there also the applicability of this method to the spaces of functions of restricted growth.

I am very indebted to the referee for his valuable remarks, especially for pointing out the importance of Buchsbaum's paper [4] on the generalized Koszul complex; this contributed to the present form of this paper. I am also indebted to M. Jarnicki for stimulating discussions on the subject and to Justyna Stankiewicz for important suggestions.

2. Proof of Theorem 1

We consider first the more complicated case $p < q$. Given $r = 0, 1, \ldots$ and $s = 1, 2, \ldots$, denote by L_r^s the space of all tuples $u = (u_J)_J$, where every J has the form $J = (j_1, \ldots, j_s)$ with $1 \le j_1, \ldots, j_s \le q$, for every J, u_J and $\bar{\partial} u_J$ are $(0, r)$ (respectively $(0, r+1)$) forms with coefficients smooth in D and continuous in $\bar{D}$, and $(u_J)_J$ is skew-symmetric with respect to J (see [7], [19]). Let L_r^0 be the space of all

$(0,r)$-forms u such that u and $\bar{\partial}u$ have coefficients smooth in D and continuous in $\bar{D}$. Set also $L_r^{-1} = 0$.

Given an open subset V of D, let $M_r^s(V)$ denote the space of all $u \in L_r^s$ which vanish on V.

For $r = 0,1,\ldots,$ and $s = 1,2,\ldots,$ define the operators

$$P^{s,r} = (P_1^{s,r},\ldots,P_p^{s,r}) : L_r^s \to (L_r^{s-1})^p$$

in the following way: If $j_1,\ldots,j_{s-1}$ and j are integers between 1 and q, $J = (j_1,\ldots,j_{s-1})$, and $u = (u_I)_I \in L_r^s$, then for $i = 1,\ldots,p$, we put (see [7], [19])

$$(P_i^{s,r}u)_J = \sum_{j=1}^q g_{ij}u_{Jj}. \tag{5}$$

(Here $Jj = (j_1,\ldots,j_{s-1},j)$.) Set $P^{0,r} = 0$. We define also the operators $\bar{\partial}_{s,r} : L_r^s \to L_{r+1}^s$ coefficientwise, i.e. $\bar{\partial}_{s,r}u = (\bar{\partial}u_I)_I$. We will usually write P, P_i and $\bar{\partial}$ instead of $P^{s,r}$, $P_i^{s,r}$ and $\bar{\partial}_{s,r}$. Note that $P_iP_i = 0$, $\bar{\partial}\bar{\partial} = 0$ and $\bar{\partial}P_i = P_i\bar{\partial}$.

If $J = (j_1,\ldots,j_s)$ with $1 \leq j_1,\ldots,j_s \leq q$, we set $J\backslash j_l =: (j_1,\ldots,j_{l-1},j_{l+1},\ldots,j_s)$.

We need some auxiliary results. The first lemma is a counterpart of [19], Lemma 2.

Lemma 1. *Let V be a neighborhood of w in D, and suppose that the functions $g_{ij} \in A(D)$, $i = 1,\ldots,p$, $j = 1,\ldots,q$, satisfy* (3) *for every $z \in \bar{D}\backslash\{w\}$. Let $u \in M_r^s(V)$ be such that $Pu = 0$. Then there exists $v \in M_r^{s+1}(V)$ such that for every $i = 1,\ldots,p$, $P_iv = u$.*

Proof. Denote by N the set of all multiindices $K = (k_1,\ldots,k_p)$ with $1 \leq k_1 < \cdots < k_p \leq q$. Given $K = (k_1,\ldots,k_p) \in N$, let g_K be the $(p \times p)$-minor of the matrix $[g_{ij}]$, consisting of columns with numbers $k_1,\ldots,k_p$. Set $Z_K = \{z \in \bar{D} \mid g_K(z) = 0\}$. There exists a family $\{\varphi_K\}_{K\in N}$ of functions smooth in $\mathbf{C}^n\backslash\{w\}$, such that for every K, $0 \leq \varphi_K \leq 1$, φ_K vanishes in some neighborhood of $Z_K\backslash\{w\}$ in $\mathbf{C}^n\backslash\{w\}$, and $\sum_{K\in N}\varphi_K \equiv 1$ in $\mathbf{C}^n\backslash\{w\}$. Let W be a neighborhood of w relatively compact in V. Choose a smooth cut-off function φ such that φ vanishes in W and $\varphi \equiv 1$ outside of V. For every $K \in N$, define a function ψ_K in $\bar{D}$ by

$$\psi_K = \frac{\varphi\varphi_K}{g_K}$$

in $\bar{D}\backslash Z_K$ and by zero otherwise. Then ψ_K is smooth in D and continuous in $\bar{D}$.

We define the tuple $v = (v_J)_J$ from the assertion in the following way: Given $J = (j_1,\ldots,j_{s+1})$ with $1 \leq j_1,\ldots,j_{s+1} \leq q$, set

$$v_J = \sum_{l=1}^{s+1}(-1)^{s+1-1}\sum \psi_K \det \begin{bmatrix} g_{1k_1}, \dots, g_{1k_{t-1}}, u_{J\setminus j_l}, g_{1k_{t+1}}, \dots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \dots, g_{pk_{t-1}}, u_{J\setminus j_l}, g_{pk_{t+1}}, \dots, g_{pk_p} \end{bmatrix}$$

where the inner summation is extended over all pairs (K,t) such that $K = (k_1, \dots, k_p) \in N$ and $j_l = k_t$ for some t between 1 and p. (For $p = 1$, this is the formula from [19], Lemma 2.) Then $v = (v_J)_J \in M_r^{s+1}(V)$. We claim that for every $i = 1, \dots, p$, $P_i v = u$, i.e. for every $I = (i_1, \dots, i_s)$ with $1 \le i_1, \dots, i_s \le q$, the formula

$$\sum_{j=1}^{q} g_{ij} v_{Ij} = u_I$$

holds.

By definition,

$$\sum_{j=1}^{q} g_{ij} v_{Ij} = \sum_{j=1}^{q} g_{ij} \sum_{l=1}^{s+1}(-1)^{s+l-1}\sum \psi_K$$

$$\times \det \begin{bmatrix} g_{1k_1}, \dots, g_{1k_{t-1}}, u_{Ij\setminus(Ij)_l}, g_{1k_{t+1}}, \dots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \dots, g_{pk_{t-1}}, u_{Ij\setminus(Ij)_l}, g_{pk_{t+1}}, \dots, g_{pk_p} \end{bmatrix}, \tag{6}$$

where we have set $(Ij)_l = i_l$ for $l = 1, \dots, s$ and $(Ij)_l = j$ for $l = s+1$, and the inner summation is extended over all pairs (K,t) with $K = (k_1, \dots, k_p) \in N$ and $(Ij)_l = k_t$ for some $1 \le t \le p$. Separating the terms with $l = 1, \dots, s$ and $l = s + 1$, in the right-hand side of (6), we obtain the expression

$$\sum_{l=1}^{s}(-1)^{s+l-1}\sum \psi_K \sum_{j=1}^{q} g_{ij} \det \begin{bmatrix} g_{1k_1}, \dots, g_{1k_{t-1}}, u_{(I\setminus i_l)j}, g_{1k_{t+1}}, \dots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \dots, g_{pk_{t-1}}, u_{(I\setminus i_l)j}, g_{pk_{t+1}}, \dots, g_{pk_p} \end{bmatrix}$$

$$+ \sum_{j=1}^{q} g_{ij} \sum \psi_K \det \begin{bmatrix} g_{1k_1}, \dots, g_{1k_{t-1}}, u_I, g_{1k_{t+1}}, \dots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \dots, g_{pk_{t-1}}, u_I, g_{pk_{t+1}}, \dots, g_{pk_p} \end{bmatrix}. \tag{7}$$

The second summation in the first term is extended over all (K,t) with $K \in N$ such that $i_l = k_t$ for some $1 \le t \le p$, and the inner summation in the second term holds over all (K,t) with $K \in N$ and $j = k_t$ for some t.

Consider the first term in (7). If $s = 0$, this term does not exist at all. If $s > 0$, we could change the summation in (6) since t does not depend on j there. Since for every $i = 1, \ldots, p$, $\sum_{j=1}^{q} g_{ij} u_{(I \setminus i_l)_j} = P_i u_{I \setminus i_l} = 0$ by assumption, this term vanishes. The second term in (7) can be written as

$$\sum_{K \in N} \psi_K \left(\sum_{m=1}^{p} g_{ik_m} \det \begin{bmatrix} g_{ik_1}, \ldots, g_{1k_{m-1}}, 1, g_{1k_{m+1}}, \ldots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \ldots, g_{pk_{m-1}}, 1, g_{pk_{m+1}}, \ldots, g_{pk_p} \end{bmatrix} \right) u_I.$$

By elementary linear algebra, the inner sum equals g_K, and so this term is equal to $\sum_{K \in N} \varphi \varphi_K u_I = u_I$ (we use here the fact that u_I vanishes in V).

To complete the cohomological argument we need two further results, which are analogues of Lemmas 1′ and 3 from [19].

LEMMA 2. *Let V be a neighborhood of w in D. Suppose that the functions $g_{ij} \in A(D)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, satisfy* (3) *for every $z \in \bar{D} \setminus \{w\}$. Fix $r \geq 2$, and $s \geq 0$, and suppose that $u \in M_r^s(V)$ is such that $\bar{\partial} u = 0$ and $P_1 u = \cdots = P_p u$. Then there exists a neighborhood W of w contained in V, and $v \in M_{r-1}^s(W)$ such that $\bar{\partial} v = u$ and*

$$P_1 v = \cdots = P_p v. \tag{8}$$

(Note that the similar result does not hold in general for $r = 1$.)

LEMMA 3. *Let V and g_{ij} be as in Lemma* 2. *Fix $r \geq 1$ and $s \geq 0$, and let $u \in M_r^s(V)$ be such that $\bar{\partial} u = 0$ and $Pu = 0$. Then there exist a neighborhood W of w with $W \subset V$, and $v \in M_r^{s+1}(W)$ such that $\bar{\partial} v = 0$ and $P_1 v = \cdots = P_p v = u$.*

We prove both lemmas simultaneously, by induction on p. For $p = 1$, they are essentially in [19]; because we need here some modifications in comparison to [19], even for $p = 1$, we include the proof here, for the convenience of the reader.

Let u be as in Lemma 2. By [5] or [16], there exists $u_1 \in L_{r-1}^s$ such that $\bar{\partial} u_1 = u$. Since u vanishes in V, u_1 is smooth and $\bar{\partial}$-closed there. Take two balls B and W centered at w such that $\overline{W} \subset B \subset V$. By [8] there exists a tuple $t = (t_I)_I$ of smooth $(0, r-2)$-forms in V such that $\bar{\partial} t = u_1$. Let φ be a smooth cut-off function such that supp $\varphi \subset B$ and $\varphi \equiv 1$ on $\overline{W}$. Then $v =: u_1 - \bar{\partial}(\varphi t)$ satisfies the assertion.

In order to prove Lemma 3 for $p = 1$, we proceed like in [19], by downward induction on r. The lemma is clearly true for $r > n$. So suppose the lemma to be proved for $r + 1$ and for every s, and let u be as in the assumption. By Lemma 1, there exists $u_1 \in M_r^{s+1}(V)$ such that $Pu_1 = u$. Then $\bar{\partial} u_1 \in M_{r+1}^{s+1}(V)$ is $\bar{\partial}$-closed and $P \bar{\partial} u_1 = \bar{\partial} P u_1 = 0$, so by the induction assumption there exists a neighborhood

T of w in D with $T \subset V$, and $t \in M_{r+1}^{s+2}(T)$ such that $\bar\partial t = 0$ and $Pt = \bar\partial u_1$. By Lemma 2, there exists a neighborhood W of w, $W \subset T$, and $t_1 \in M_r^{s+2}(W)$ such that $\bar\partial t_1 = t$. Then $u =: u_1 - Pt_1$ has the desired properties.

Now assume both lemmas to be proved for all $(p' \times q)$-matrices with p' not greater than some $p < q$. Suppose that the $((p+1) \times q)$-matrix $[g_{ij}]$ and $u \in M_r^s(V)$ satisfy the assumptions of Lemma 2. If $s = 0$, then all operators P_i are zero by definition; therefore if a neighborhood W of w with $W \subset V$ and $v \in M_{r-1}^0(W)$ such that $\bar\partial v = u$ are constructed as in the proof of Lemma 2 for $p = 1$, then v satisfies (8) trivially. Suppose therefore that $s > 0$. Note that both $(p \times q)$-matrices $[g_{ij}]$ and $[k_{ij}] =: [g_{ij} - g_{p+1,j}]$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, satisfy (3) with p for every $z \in \bar D \setminus \{w\}$. By Lemma 2 for p, there exists a neighborhood T of w with $T \subset V$ and $u_1 \in M_{r-1}^s(T)$ such that $\bar\partial u_1 = u$, and $P_1 u_1 = \cdots = P_p u_1$. For $i = 1, \ldots, p$, define the operators R_i similarly as in (5), but with respect to the functions k_{ij} instead of g_{ij}. Then $R_i = P_i - P_{p+1}$, $i = 1, \ldots, p$, and $R_1 u_1 = \cdots = R_p u_1$. Therefore $t =: R_1 u_1 = \cdots = R_p u_1$ is a well-defined element of $M_{r-1}^{s-1}(T)$, and moreover $\bar\partial t = R_1 \bar\partial u_1 = P_1 u - P_{p+1} u = 0$ and $R_i t = R_i R_i u_1 = 0$, $i = 1, \ldots, p$. By Lemma 3 for p, there exists a neighborhood W of w with $W \subset T$, and $t_1 \in M_{r-1}^s(W)$, such that $\bar\partial t_1 = 0$ and $R_1 t_1 = \cdots = R_p t_1 = t$. Then $v =: u_1 - t_1 \in M_{r-1}^s(W)$ satisfies the assertion of Lemma 2 for the $((p+1) \times q)$-matrix $[g_{ij}]$.

To prove Lemma 3 for $p+1$ we proceed, as before, by downward induction on r, the assertion being trivially satisfied for $r > n$. Therefore assume the lemma to be proved for $r+1$ and for every s, and let $u \in M_r^s(V)$ be as in the assumption. By Lemma 1, there exists $u_1 \in M_r^{s+1}(V)$ such that $P_1 u_1 = \cdots = P_{p+1} u_1 = u$ (but u_1 need not be $\bar\partial$-closed). The tuple $\bar\partial u_1 \in M_{r+1}^{s+1}(V)$ satisfies the induction assumption, and so there exists a neighborhood T of w such that $T \subset V$, and $t \in M_{r+1}^{s+2}(T)$ with $\bar\partial t = 0$ and $P_1 t = \cdots = P_{p+1} t = \bar\partial u_1$. By Lemma 2 for $p+1$, we can find a neighborhood W of w, $W \subset T$, and some $t_1 \in M_r^{s+2}(W)$, such that $\bar\partial t_1 = t$ and $P_1 t_1 = \cdots = P_{p+1} t_1$. Setting $t_2 =: P_1 t_1 = \cdots = P_{p+1} t_1$ and $v = u_1 - t_2$, we obtain the desired tuple $v \in M_r^{s+1}(W)$.

To proceed with the proof of the theorem (still with $p < q$) we need two other auxiliary results: the first one can be treated as an analogue of the theorem for tuples in L_0^s with $s > 0$, the second is a counterpart of Lemma 2 for $r = 1$.

PROPOSITION 4. *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$, and w a fixed point in D. Suppose that the functions $g_{ij} \in A(D)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, satisfy* (3) *for every $z \in \bar D \setminus \{w\}$. Let $u = (u_I)_I \in L_0^s$, $s \geq 0$, be such that for each I, $u_I \in A(D)$, and $Pu = 0$ (with P defined by* (5)*). Suppose also that there exists a neighborhood U of w in D and a tuple $\tilde v = (\tilde v_J)_J$ such that for ev-*

ery $J = (j_1, \ldots, j_{s+1})$, $\tilde{v}_J$ is holomorphic in U, and $P_i\tilde{v} = u$ in U, $i = 1, \ldots, p$. Then there exists $v = (v_J)_J \in L_0^{s+1}$ such that for each J, $v_J \in A(D)$, and $P_i v = u$ in $\bar{D}$, $i = 1, \ldots, p$. (In other words, the local divisibility of u by a matrix $[g_{ij}]$ with holomorphic factors implies the global divisibility.)

LEMMA 5. *If D and w are as above, W is a neighborhood of w in D, the $(p \times q)$-matrix $[g_{ij}]$ of functions from the algebra $A(D)$ satisfies the assumption* (3) *for every $z \in \bar{D} \setminus \{w\}$, and $v \in M_1^s(W)$ is such that $\bar{\partial} v = 0$ and $P_1 v = \cdots = P_p v$, then there exists $t \in L_0^s$ such that $\bar{\partial} t = v$ and $P_1 t = \cdots = P_p t$.*

We show first, similarly as in [19], that under the assumptions of either Theorem 1 or Proposition 4, the global division problem can be solved differentiably.

Consider the situation as described in the assumption of Theorem 1. We will prove that there exist functions $\hat{h}_1, \ldots, \hat{h}_q$ which are smooth in D, continuous in $\bar{D}$, and holomorphic in some neighborhood V of w, such that for every $i = 1, \ldots, p$,

$$f_i = \textstyle\sum_{j=1}^{q} g_{ij} \hat{h}_j \tag{9}$$

in $\bar{D}$. Choose a smooth cut-off function φ_0 with support in U, such that $\varphi_0 \equiv 1$ in some neighborhood of w, relatively compact in U. Let N, $K \in N$, g_K, Z_K and φ_K have the same meaning as in the proof of Lemma 1. If $K = (k_1, \ldots, k_p)$, there exist functions $\tilde{h}_{K,k_1}, \ldots, \tilde{h}_{K,k_p}$, uniquely determined in $\bar{D} \setminus Z_K$, such that $f_i = \sum_{l=1}^{p} g_{ik_l} \tilde{h}_{K,k_l}$. Moreover, the functions $\tilde{h}_{K,k_1}$ are smooth in $D \setminus Z_K$ and continuous in $\bar{D} \setminus Z_K$, and the functions $(1 - \varphi_0)\varphi_K \tilde{h}_{K,k_l}$ extend (by zero) to the functions h_{K,k_l} smooth in D, continuous in $\bar{D}$, and vanishing in V. For $j \neq k_1, \ldots, k_p$, set $h_{K,j} \equiv 0$. Then the functions

$$\hat{h}_j =: \varphi_0 \tilde{h}_j + \sum_{K \in N} h_{K,j}, \qquad j = 1, \ldots, q,$$

satisfy (9), and have other desired properties. (For $p = 1$, the above construction is in [19].)

If the situation is as in the assumption of Proposition 4, write $u = \varphi_0 u + (1 - \varphi_0)u$, with φ_0 as above. Extend $\varphi_0 \tilde{v}$ by zero to all of $\bar{D}$; then $P_i(\varphi_0 \tilde{v}) = \varphi_0 u$ in $\bar{D}$ for $i = 1, \ldots, p$. Moreover, $(1 - \varphi_0)u \in M_0^s(V)$ and $P(1 - \varphi_0)u = (1 - \varphi_0)Pu = 0$, so by Lemma 1 there exists $t \in M_0^{s+1}(V)$ such that for $i = 1, \ldots, p$, $P_i t = (1 - \varphi_0)u$. It follows that $\hat{v} =: \varphi_0 \tilde{v} + t$ is an element of L_0^{s+1} such that for every J, $\hat{v}_J$ is smooth in D, continuous in $\bar{D}$, and holomorphic in some neighborhood W of w, and $P_i \hat{v} = u$ for $i = 1, \ldots, p$; thus the division problem from Proposition 4 also can be solved differentiably.

Now we are able to prove Theorem 1, Proposition 4 and Lemma 5. We do this simultaneously, by induction on p.

For $p = 1$, this is done in [19] (at least for $s = 0$). Suppose all the results to be valid for p. Let the $((p+1) \times q)$-matrix $[g_{ij}]$ satisfy (3) at every point $z \in \bar{D} \setminus \{w\}$. Consider the situation as in the assumptions of Theorem 1. Let $\hat{h}_1, \ldots, \hat{h}_q$ be the functions satisfying (9). Since they are holomorphic in some neighborhood V of w, the tuple $\bar{\partial}\hat{h}$ is in $M_1^1(V)$; it is moreover $\bar{\partial}$-closed and $P_i\bar{\partial}\hat{h} = \bar{\partial}P_i\hat{h} = \bar{\partial}f_i = 0$, $i = 1, \ldots, p+1$. Thus by Lemma 3, there exists a neighborhood W of w with $W \subset V$, and $v \in M_1^2(W)$ such that $\bar{\partial}v = 0$ and $P_1 v = \cdots = P_{p+1}v = \bar{\partial}\hat{h}$. Since the $(p \times q)$-matrix $[g_{ij}]$ satisfies (3) for every $z \in \bar{D} \setminus \{w\}$ with p, it follows from Lemma 5 for p that $v = \bar{\partial}v_1$ for some $v_1 \in L_0^2$ such that $P_1 v_1 = \cdots = P_p v_1$. We proceed now similarly as in the proof of Lemma 2. Considering the $(p \times q)$-matrix $[k_{ij}] =: [g_{ij} - g_{p+1,j}]$ and the operators $R_i = P_i - P_{p+1}$, we see that $u =: R_1 v_1 = \cdots = R_p v_1$ is a well-defined element of L_0^1 such that $\bar{\partial}u = 0$ and $R_i u = 0$. Moreover, the matrix $[k_{ij}]$ satisfies (3) for all $z \in \bar{D} \setminus \{w\}$ and the tuple v_1 is holomorphic in W, since $\bar{\partial}v_1 = v$ and v vanishes in W; this means that the tuple u satisfies the assumptions of Proposition 4 for p, for the operator R defined by the functions k_{ij}, and for $\tilde{v} =: v_{1|W}$. Hence there exists the tuple $t \in L_0^2$, consisting of functions from the algebra $A(D)$, such that $R_i t = u$ in $\bar{D}$ for $i = 1, \ldots, p$. If we now set $t_1 =: v_1 - t$, it follows that $\bar{\partial}t_1 = v$ and that (like at the end of the proof of Lemma 2) $t_2 =: P_1 t_1 = \cdots = P_{p+1}t_1$ is a well-defined element of L_0^1. Then $h =: h - t_2$ satisfies the assertion of the theorem.

The proof of Proposition 4 for $p + 1$ is quite similar.

It remains to show that Lemma 5 holds for $p + 1$. The assertion is trivial for $s = 0$, since all operators P_i are then zero, and the result follows from [5] or [16]. Assume therefore that $s > 0$ and v is as in the assumption. Similarly as above, by the induction assumption there exists $v_1 \in L_0^s$ such that $\bar{\partial}v_1 = v$ and $P_1 v_1 = \cdots = P_p v_1$. Then the tuple $u =: R_i v_1 = P_i v_1 - P_{p+1}v_1$ is a well-defined element of L_0^{s-1}, and $\bar{\partial}u = R_i u = 0$, $i = 1, \ldots, p$. Moreover, the functions in the tuple v_1 are holomorphic in W, because $\bar{\partial}v_1 = v$ and v vanishes in W by assumption. Hence, by Proposition 4, there exists $u_1 \in L_0^s$ consisting of functions from $A(D)$, such that $u = R_i u_1$ for $i = 1, \ldots, p$. Setting $t =: v_1 - u_1$, we check similarly, as in the proof of Lemma 2, that $\bar{\partial}t = v$ and $P_1 t = \cdots = P_{p+1}t$.

Consider now the case $p \geq q$. This case is quite easy, and it turns out that the functions $\tilde{h}_j$ simply continue to the functions $h_j \in A(D)$, satisfying (2) in $\bar{D}$. In fact, define N to be the set of all tuples $K = (k_1, \ldots, k_q)$ with $1 \leq k_1 < \cdots < k_q \leq p$. For $K \in N$, let g_K be the $(q \times q)$-minor of the matrix $[g_{ij}]$, consisting of rows with numbers $k_1, \ldots, k_q$, and set $Z_K = \{z \in \bar{D} | g_K(z) = 0\}$. If $K =$

$(k_1,\ldots,k_q)\in N$ and g_K does not vanish identically in $\bar D$, then, by Cramer's rule, there exist uniquely determined functions $h_{K,1},\ldots,h_{K,p}$, holomorphic in $D\backslash Z_K$ and continuous in $\bar D\backslash Z_K$, such that in $\bar D\backslash Z_K$

$$f_{k_l}=\textstyle\sum_{j=1}^{q} g_{k_lj}h_{K,j}\qquad\text{for } l=1,\ldots,p. \tag{10}$$

By assumption, similar relations hold in some neighborhood U of w, but with functions $\tilde h_j$ instead of $h_{K,j}$. But then $U\backslash Z_K$ is a non-void open subset of U, and both relations (4) and (10) hold simultaneously in $U\backslash Z_K$, so by the uniqueness of the functions $h_{K,j}$, mentioned above, $h_{K,j}=\tilde h_j$ in $U\backslash Z_K$. It follows that if $K,L\in N$ are such that g_K and g_L do not vanish identically, then $h_{K,j}=h_{L,j}$ in $U\backslash(Z_K\cup Z_L)$, and since the sets $D\backslash Z_K$ and $D\backslash Z_L$ are connected, we have $h_{K,j}=h_{L,j}$ in all of $D\backslash(Z_K\cup Z_L)$ by identity principle for holomorphic functions; hence $h_{K,j}$ and $h_{L,j}$ coincide in all of $(\bar D\backslash Z_K)\cap(\bar D\backslash Z_L)$. Now let $z\in\bar D$. If $z\neq w$, then by (3) there exists $K\in N$ such that $g_K(z)$ is non-singular; for $j=1,\ldots,q$, define then $h_j(z)=:h_{K,j}(z)$. For $z=w$, set $h_j(z)=\tilde h_j(z)$. It is clear from the above considerations that the functions $h_1,\ldots,h_q$ are well-defined and that they satisfy (2) in $\bar D$; moreover, they are extensions of functions $\tilde h_j$, as claimed above. This ends the proof of the theorem for the case $p\geq q$.

3. Other results and problems

The cohomological device, presented in the previous section, together with results on the solution on the $\bar\partial$-problem, gives "local-to-global" matrix division theorems in many cases, for which the decomposition of the form $f=\sum_{i=1}^{N} g_if_i$ is known. We present here two results of this type. As mentioned above, the cohomological part remains the same as in Section 2; the eventual changes concern the construction of convenient neighborhoods of common zero sets of the functions g_{ij}, and of some cut-off functions.

If D is a domain in $\mathbf{C}^n$ and E an open subset of ∂D, denote by $\tilde A_E(D)$ any one of the following spaces:

$A_E^k(D)=:\{f$ holomorphic in $D\,|\,f$ extends continuously together with all its derivatives up to order k to the set $D\cup E\}\}$, $k=0,1,\ldots,$

$H_E^{\infty,k}(D)=:\{f\in A_E^{k-1}(D)\,|$ the derivatives of f of order k are bounded on every compact subset of $D\cup E\}$, $k=0,1,\ldots,$

$(\Lambda_t)_E(D)=:\{f\in A_E^k(D)$ with $k<t<k+1\,|$ the derivatives of f of order k satisfy Hölder condition of order $t-k$ on every compact subset of $D\cup E\}$ (here t is a positive real number which is not an integer).

THEOREM 2 (cf. [17], [10]). *Let D be a domain of holomorphy in $\mathbf{C}^n$, and suppose that E is an open subset of ∂D such that ∂D is smooth and strictly pseudoconvex in the neighborhood of every point of E. Let $Z = \{z_k\}$ be a (finite or infinite) sequence of points in D without accumulation point in $D \cup E$. Suppose that the functions f_i and g_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, from the algebra $\tilde{A}_E(D)$ are such that there exists a neighborhood W of Z and functions $\tilde{h}_1, \ldots, \tilde{h}_q$ holomorphic in W, for which the decomposition (2) holds in W, and for every $z \in (D \cup E)\setminus Z$, the condition (3) is satisfied.*

Then there exist functions $h_1, \ldots, h_q \in \tilde{A}_E(D)$ such that (2) holds in all of $D \cup E$.

THEOREM 3 (cf. [11]). *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ and suppose that the functions f_i and g_{ij} from the algebra $A_Q(D \times D)$ (where $Q = \partial(D \times D)\setminus\Delta(\partial D)$ with $\Delta(\partial D) = \{(z,z) \mid z \in \partial D\}$) are such that the local decomposition (2) holds in some neighborhood W of $\Delta(D)$ in $D \times D$, and that (3) is satisfied in all points of $(\bar{D} \times \bar{D})\setminus\Delta(\bar{D})$. Then (2) holds globally with some functions $h_j \in A_Q(D \times D)$.*

The convenient results on the solution of the $\bar{\partial}$-problem, necessary in the proof of Theorem 2, are contained in [17] and in [10], and those for Theorem 3 are in [11]. Theorem 3 can be also proved in a more general situation – see [12]. One can check that the above results hold also for domains in complex submanifolds of $\mathbf{C}^n$.

As mentioned in the introduction, the use of the Koszul complex in the present setting (for $p = 1$) was introduced in [7] in order to study the spaces of holomorphic functions with restricted growth. If D is a domain in $\mathbf{C}^n$ and p a non-negative function, let $A_p(D)$ be the set of all analytic functions in D such that for some constants C_1 and C_2

$$|f(z)| \leq C_1 \exp(C_2 p(z)), \qquad z \in D.$$

$A_p(D)$ is a ring (see [7]). Hörmander obtained the following theorem on the generators of $A_p(D)$:

THEOREM ([7], Thm 1). *Let p be a non-negative plurisubharmonic function in the domain of holomorphy $D \subset \mathbf{C}^n$ such that:*

(i) *All polynomials belong to $A_p(D)$.*

(ii) *There exist constants $K_1, \ldots, K_4$ such that $z \in D$ and $|z - \zeta| \leq \exp(-K_1 p(z) - K_2)$ imply that $\zeta \in D$ and $p(\zeta) \leq K_3 p(z) + K_4$.*

Then $f_1, \dots, f_N \in A_p(D)$ generate $A_p(D)$ if and only if there exist positive constants c_1 and c_2 such that

$$|f_1(z)| + \cdots + |f_N(z)| \geq c_1 \exp(-c_2 p(z)), \qquad z \in D. \tag{11}$$

The generalization of Hörmander's theorem to the matrix case was proved by Kelleher and Taylor ([13], Corollary 4.4). By means of the generalized Koszul complex the authors obtained there the necessary and sufficient conditions on the matrix $[g_{ij}]$ of functions $g_{ij} \in A_p(D)$ in order that for every choice $f_1, \dots, f_p$ of functions from $A_p(D)$ the decomposition (2) holds with some $h_j \in A_p(D)$. By means of the cohomological method described above and of the results on the solution of the $\bar{\partial}$-equation with bounds from [7] we could prove only a following sufficient condition:

THEOREM 4. *Let D and p be as above. Suppose that the functions $g_{ij} \in A_p(D)$, $i = 1, \dots, p$, $j = 1, \dots, q$, are such that there exist some positive constants c_1 and c_2 with*

$$\sum_{K \in N} |g_K(z)| \geq c_1 \exp(-c_2 p(z)), \qquad z \in D, \tag{12}$$

where K and N have the same meaning as in the proof of Theorem 1.

Then for every choice $f_1, \dots, f_p$ of functions from $A_p(D)$ there exist functions $h_1, \dots, h_q \in A_p(D)$ such that

$$f_i = \textstyle\sum_{j=1}^q g_{ij} h_j$$

in D, $i = 1, \dots, p$.

The proof goes by a series of three lemmas, which correspond to Lemmas 1, 2 and 3 from Section 2, and to Lemmas 5 and 6 and Theorem 7 from [7]. The only moment which needs explication is the necessary change in the proof of the analogue of Lemma 1 from this note. Set now L_r^s to be the space of all tuples $u = (u_J)_J$, skew-symmetric with respect to J, such that for every J, u_J is a $(0, r)$-form with coefficients measurable in D, satisfying

$$\int_D |u_J|^2 e^{-2Kp} \, dm < \infty$$

for some K (see [7], p. 945 — dm denotes here the Lebesgue measure in $\mathbf{C}^n$). The above-mentioned analogue of Lemma 1 states that given $g_{ij} \in A_p(D)$ satisfying (12) and $u \in L_r^s$ such that $Pu = 0$ (with P defined by g_{ij} as in (5)) there exists $v \in$

L_r^{s+1} such that $P_i v = u$ for $i = 1, \ldots, p$. The tuple $v = (v_J)_J$ (for the case $p < q$) is now defined by the formula

$$v_J = \frac{1}{\sum\limits_{K \in N} |g_K|^2} \sum_{l=1}^{s+1} (-1)^{s+l-1} \sum \overline{g_K} \det \begin{bmatrix} g_{1k_1}, \ldots, g_{1k_{t-1}}, u_{J\setminus j_l}, g_{1k_{t+1}}, \ldots, g_{1k_p} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ g_{pk_1}, \ldots, g_{pk_{t-1}}, u_{J\setminus j_l}, g_{pk_{t+1}}, \ldots, g_{pk_p} \end{bmatrix},$$

where the inner summation holds, as before, over all pairs (K, t) with $K = (k_1, \ldots, k_p) \in N$ and $j_l = k_t$ for some $1 \leq t \leq p$. (For $p = 1$, we obtain the formula from the proof of Lemma 6 in [7].)

One could also try to prove some local-to-global division theorems for the case when the common zero-set of functions g_{ij} is positive-dimensional, instead of zero-dimensional sets $\{w\}$ or $\{z_k\}$ from Theorems 1 and 2. Suppose that D is a domain of holomorphy with sufficiently smooth boundary and that the functions f_i and g_{ij} are holomorphic in D and have some boundary regularity on ∂D, and vanish on some analytic subset M of D. Assume also that the decomposition (2) holds in some neighborhood V of M in D, with some holomorphic functions $\tilde{h}_j$. Under what conditions on f_i, g_{ij} and $\tilde{h}_j$ can one obtain the global decomposition (2) in all of $\bar{D}$ with holomorphic functions h_j, being as regular on ∂D as possible? It is easy to see that the previously described cohomological device works in the following case: D is a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ and $M = M' \cap D$, where M' is a complex submanifold of some neighborhood D' of $\bar{D}$, and $V = D \cap V'$, where V' is some neighborhood of $M' \cap \bar{D}$ in $\mathbf{C}^n$; the functions f_i and g_{ij} are, say, in some algebra $\tilde{A}(D)$ and the functions $\tilde{h}_j$ have on the set $V' \cap \bar{D}$ the boundary regularity like that described in the definition of the given algebra $\tilde{A}(D)$. Assuming then that the functions g_{ij} satisfy (3) in all points of $\bar{D} \cap V'$, one obtains the decomposition (2) in the whole set $\bar{D}$ with some functions $h_j \in \tilde{A}(D)$. This case is, however, not of great interest; it seems that the most interesting case arises when one assumes that the domain V does not touch ∂D, i.e. no regularity assumptions on the behavior of the functions $\tilde{h}_j$ on ∂D are supposed. In [3] Bonneau, Cumenge and Zériahi proved that if D is a strictly pseudoconvex domain in $\mathbf{C}^n$ with sufficiently smooth boundary, given by a defining function ρ, and the functions $g_1, \ldots, g_q \in A^m(D)$ $(q < n)$ are such that

$$dg_1(z) \wedge \cdots \wedge dg_q(z) \neq 0 \tag{13}$$

for every $z \in X' =: \{z \in \bar{D} | g_1(z) = \cdots = g_q(z) = 0\}$, and

$$\partial\rho(z) \wedge \partial g_1(z) \wedge \cdots \wedge \partial g_q(z) \neq 0$$

for every $z \in X' \cap \partial D$, then every function $f \in \Lambda_t(D)$ with $[t] \leq m - 2$, vanishing on X', admits a decomposition

$$f = \sum_{i=1}^{q} g_i h_i$$

with some functions $h_i \in \Lambda_{t-1/2}(D)$ (the last of 1/2 is here the lowest possible and cannot be avoided in general). It would be interesting to prove the local-to-global analogue of the above result for the matrix case. We remark only that the local-to-global decomposition does not hold in general without, e.g., the assumption of type (13), as the following trivial example shows:

EXAMPLE. Let D be a unit ball in $\mathbf{C}^2$. Set $f = gh$ *with* $g(z_1, z_2) = z_2^n$ and $h(z_1, z_2) = F(z_1)$, where $F \in H^\infty(U)$, but $F \notin A(U)$. (Here U is the unit disc in $\mathbf{C}$.) Then $D_1^k f = z_2^n D^k F$, and $D_2^k f = n(n-1)\cdots(n-k+1)z_2^{n-k}F$. Of course, the local as well as the global decomposition of f with respect to g holds, and the only possible factor is the function h itself. However, the function h is not even in $A(D)$, although f can be as regular on ∂D, as we wish. In fact, for every positive integer k, the functions $D_1^k f$ and $D_2^k f$ are continuous in $Q =: \bar{D} \setminus \{(z_1, 0) \mid |z_1| = 1\}$, and they vanish at $(z_1, 0)$ for $z_1 \in \bar{U}$. Since F is bounded on U, it is easy to see that $D_2^k f$ is continuous in all of $\bar{D}$. On the other hand, by Cauchy's inequalities,

$$|D^k F(z_1)| \leq \frac{\|F\|_\infty}{(1 - |z_1|)^k}$$

and for $(z_1, z_2) \in D$, $|z_2|^2 < 1 - |z_1|^2 < 2(1 - |z_1|)$, so $|D_2^k f(z_1, z_2)| \leq c \|F\|_\infty (1 - |z_1|)^{(n-3k)/2}$ with some $c > 0$, which shows that $D_2^k f$ is also continuous in all of $\bar{D}$ provided that $n > 3k$. Note that the condition (13) for the function g is not satisfied.

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